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# A Simple Example of Little Big Set

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John K. Williams

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Loosely speaking, the Hausdorff dimension of a set is the right place to measure that set. For  $s$  less than the Hausdorff dimension, the Hausdorff  $s$ -measure of the set is infinite and for  $s$  larger than the Hausdorff dimension the Hausdorff  $s$ -measure is zero. This note describes a simple example of a set with Hausdorff  $s$ -measure 0, little, and Hausdorff dimension  $s$ , big. Specifically we will construct a set with Hausdorff dimension 1 and Hausdorff 1-measure 0 and then show how it can be generalized to give such a set for any  $s$ .

To get a feel for Hausdorff dimension let's look at an example, the Cantor set. The Cantor set is defined as follows. Let  $E_0 = [0, 1]$  and then define  $E_j$  as  $E_{j-1}$  with the open middle one third of each interval removed, i.e.  $E_1 = [0, 1/3] \cup [2/3, 1]$ ,  $E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$ . Each  $E_j$  consists of  $2^j$  intervals of length  $3^{-j}$ . Cantor's set is the set  $E = \bigcap_{j=0}^{\infty} E_j$ .

The Hausdorff 1-measure is just the same as the 1-dimensional Lebesgue measure, the length of the set. From the construction, one can see that the length of the  $E_j$  is  $2^j \times (1/3)^j$  and the length of  $E = \lim_{j \rightarrow \infty} (2/3)^j = 0$ . Now if we change how we measure the length of an interval,  $[a, b]$ , from  $|a - b|$  to  $|a - b|^s$ , then the length of each  $E_j$  becomes  $2^j \times (1/3)^{js}$ . The length of  $E = \lim_{j \rightarrow \infty} (2/3^s)^j$ . If  $s > (\ln 2 / \ln 3)$ , then the limit is 0; if  $s < (\ln 2 / \ln 3)$ , the limit is infinity; and if  $s = (\ln 2 / \ln 3)$ , the limit is 1. As we will see below, the Hausdorff dimension of this set is precisely  $(\ln 2 / \ln 3)$ . Now a definition of Hausdorff dimension [1, p. 7].

Define the diameter of a non-empty subset  $U$  of  $\mathfrak{R}^n$  as  $|U| = \sup\{|x - y| : x, y \in U\}$ . If  $E \subseteq \bigcup_i U_i$  and  $0 < |U_i| \leq \delta$  for each  $i$ , we say that  $\{U_i\}$  is a  $\delta$ -cover of  $E$ .

For  $E$  a subset of  $\mathfrak{R}^n$ ,  $s$  a non-negative number, and  $\delta > 0$  define:

$$\mathfrak{H}_{\delta}^s(E) = \inf \sum_{i=1}^{\infty} |U_i|^s,$$

where the infimum is over all (countable)  $\delta$ -covers  $\{U_i\}$  of  $E$ . Then the *Hausdorff  $s$ -dimensional outer measure of  $E$*  is defined as:

$$\mathfrak{H}^s(E) = \lim_{\delta \rightarrow 0} \mathfrak{H}_{\delta}^s(E).$$

The limit exists since  $\mathfrak{H}_{\delta}^s$  increases as  $\delta$  decreases but may be infinite. The restriction of  $\mathfrak{H}^s$  to the  $\sigma$ -field of  $\mathfrak{H}^s$ -measurable sets is called the *Hausdorff  $s$ -dimensional measure*.

For each  $E$ ,  $\mathfrak{H}^s(E)$  is non-increasing as  $s$  increases from 0 to  $\infty$  (as soon as  $\delta$  is less than 1,  $|U_i|^s$  decreases as  $s$  goes from 0 to  $\infty$ ). Also if  $s < t$ , then

$$\mathfrak{H}_{\delta}^s(E) \geq \delta^{s-t} \mathfrak{H}_{\delta}^t(E),$$

which implies that if  $\mathfrak{H}^t(E)$  is positive, then  $\mathfrak{H}^s(E)$  is infinite. There is then a unique value,  $\dim E$ , called the *Hausdorff dimension of  $E$* , such that

$$\mathfrak{H}^s(E) = \infty \text{ if } 0 \leq s < \dim E \quad \text{and} \quad \mathfrak{H}^s(E) = 0 \text{ if } \dim E < s < \infty.$$

Three observations are immediate. First, if  $E$  is a subset of  $F$  then the Hausdorff dimension of  $E$  is less than or equal to the Hausdorff dimension of  $F$ . Secondly, the Hausdorff dimension of  $\mathbb{R}^n$  is  $n$ . Putting the first two together, the Hausdorff dimension of a subset of  $\mathbb{R}^n$  is less than or equal to  $n$ . The question addressed in this note is can  $\mathfrak{H}^s(E)$  be zero when  $s$  is  $\dim E$ ?

If  $E$  is  $\mathbb{R}^1$ , then  $\dim E$  is 1 and  $\mathfrak{H}^1(E) = \infty$ . If  $E$  is a line segment of length  $l$ , then  $\dim E$  is again 1 and  $\mathfrak{H}^1(E)$  is  $l$ . Is there a set  $E$  where  $\dim E$  is 1 while  $\mathfrak{H}^1(E) = 0$ ? At first it does not seem possible. The idea of Hausdorff dimension is to find a place where the set should be measured, for  $s$  smaller than the dimension, the measure is infinite and for  $s$  larger, the measure is zero. A slight modification of the Cantor set will lead us to a set which has the desired properties.

One can view the Cantor set as the invariant set for a pair of linear contractions applied to the unit interval. If we define:

$$f_1(x) = (1/3)x \quad \text{and} \quad f_2(x) = (1/3)x + 2/3,$$

Then each  $E_j$  above may be defined inductively as  $E_0 = [0, 1]$  and  $E_j = f_1(E_{j-1}) \cup f_2(E_{j-1})$ .

The advantage of viewing the Cantor set in this manner is that we can apply a theorem of Moran [3, Thm. II] which in this context can be stated as follows:

**Theorem.** Let  $\{f_1, \dots, f_n\}$  be a set of linear contractions, each of which contracts by a factor of  $w_n$ . Let  $E_0$  be a set where  $f_j(E_0)$  and  $f_k(E_0)$  are disjoint for  $j \neq k$ ,  $E_j = \bigcup_{i=1}^n f_i(E_{j-1})$ , and  $E = \bigcap_{j=0}^{\infty} E_j$ . Then the Hausdorff dimension of  $E$  is  $s$  where  $s$  is defined by the equation:

$$\sum_{i=1}^n w_i^s = 1$$

and the Hausdorff  $s$ -measure is finite and positive.

Applying this to the Cantor set, we see that the Hausdorff dimension is  $\log 2 / \log 3$  and has positive Hausdorff  $(\log 2 / \log 3)$ -measure.

We can modify the construction of the Cantor set slightly by taking out the middle  $1/m$ th section at each step. This means modifying the linear contractions to be:

$$f_1(x) = \left(\frac{m-1}{2m}\right)x \quad \text{and} \quad f_2(x) = \left(\frac{m-1}{2m}\right)x + \frac{m+1}{2m}.$$

Applying Moran theorem we have the following theorem:

**Theorem.** The Hausdorff dimension of  $\mathcal{C}_m$  is

$$s = \log 2 / \log \left( \frac{2m}{m-1} \right)$$

and  $\mathfrak{H}^s(\mathcal{C}_m)$  is finite and positive.

Now the set we are looking for is just the union of the  $\mathcal{C}_m$ . Set  $\mathcal{C} = \bigcup_{j=3}^{\infty} \mathcal{C}_m$ .

**Theorem.** The Hausdorff dimension of  $\mathcal{C}$  is 1 and  $\mathfrak{H}^1(\mathcal{C}) = 0$ .

Since the dimension of each of the  $\mathcal{C}_m$  is less than one,  $\mathfrak{H}^1(\mathcal{C}_m) = 0$ . Therefore

$$\mathfrak{H}^1(\mathcal{C}) = \mathfrak{H}^1\left(\bigcup_{m=3}^{\infty} \mathcal{C}_m\right) \leq \sum_{m=3}^{\infty} \mathfrak{H}^1(\mathcal{C}_m) = 0.$$

But the dimension of  $\mathcal{C}$  must be greater than or equal to the dimension of each  $\mathcal{C}_m$ . Since the dimension of  $\mathcal{C}_m$  tends to 1 as  $m$  tends to  $\infty$ , the dimension of  $\mathcal{C}$  must be one.  $\square$

This construction can be generalized in two directions. First we can construct a Cantor like set of any dimension  $s$  between 0 and 1 by pulling out the middle  $\alpha$  at each stage where  $s$  and  $\alpha$  are related by:

$$\alpha = 1 - 2^{1-1/s}.$$

To find a set with dimension  $s$  and Hausdorff  $s$ -measure 0, find a sequence which converges to  $s$ ,  $\{s_i\}$ ; construct a Cantor like set for each  $s_i$ ; and form the union of these sets.

Secondly we can reach dimensions higher than 1 by pulling squares out the unit square (a Sierpinski Gasket) or pulling cubes out the unit cube in dimension three and above.

Finally, there are other sets with this property. Most notably, a Besicovitch set is a set with zero planar measure, lines in all directions and Hausdorff dimension 2. The exact description and proof of its properties is a little more difficult [1, ch. 7].

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E 402 [1940, 48]. *Proposed by Irving Kaplansky, Harvard University.*

If  $n$ ,  $r$ , and  $a$  are positive integers, the congruence  $n^2 \equiv n \pmod{10^a}$  obviously implies  $n^r \equiv n \pmod{10^a}$ . (When such a number  $n$  has only  $a$  digits, it is called an automorphic number.) For what values of  $r$  does  $n^r \equiv n \pmod{10^a}$  imply  $n^2 \equiv n \pmod{10^a}$ ?

—*American Mathematical Monthly* 47, (1940) p. 572.